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INFERENCE FOR WRAPPED SYMMETRIC α-STABLE CIRCULAR MODELS

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SUMMARY. This article provides accurate approximations for the distribution of the length of the resultant as well as for the conditional distribution for the circular mean given the resultant length, when the data come from a wrapped symmetric α-stable model. Since the latter distribution is asymptotically independent of the concentration parameter for a given value of the resultant length, it can be used for inference on the mean direction when the concentration parameter is unknown. The value of the saddlepoint methods lies in making such asymptotics available for very small sample sizes. Besides possessing important theoretical properties, this class of circular models is very rich and includes the wrapped normal and the wrapped Cauchy distributions as special cases. These distributional results allow one to employ this broader class of parametric distributions instead of the von Mises distribution, as is typically done with circular data.

1. Introduction

In many scientific disciplines observations are directions and are referred to as "directional data". In particular, two-dimensional directions, which can be represented as points on the unit circle, are called "circular data". Examples of directional data can be found in various fields: directions of remanent magnetism are sometimes used to interpret possible magnetic pole migrations during geological time, directions of migratory of birds provide relevant information in ornithology, etc. (see Fisher, 1993, Mardia and Jupp, 2000, or Jammalamadaka and SenGupta, 2001 — from here on, JS, 2001). Generally,

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any periodic phenomenon with a known period, like circadian rhythms, can be represented by a circular model. Wrapped distributions provide a rich and useful class of models for circular data. The special cases of the wrapped normal (WN) and the wrapped Cauchy, going back to Lévy (1939), are widely discussed (cf. Mardia, 1972) and our attention in this paper is on the general wrapped symmetric α-stable (WSαS) class.

If X is any continuous random variable on the real line with absolutely continuous distribution function G defining a density g, the density function f of the circular random variable, $X \mathrm{mod}(2\pi)$, is obtained by wrapping g around the circumference of a circle, i.e. by $f(\theta) = \sum_{j=-\infty}^{\infty} g(\theta+2j\pi)$, for $\theta \in [0, 2\pi)$. The corresponding distribution function would be given by $F(\theta) = \sum_{j=-\infty}^{\infty}[G(\theta + 2j\pi) - G(2j\pi)]$. Equivalently, when f belongs to $\mathcal{L}^2[0, 2\pi)$, the Hilbert space of square integrable functions on $[0, 2\pi)$, a wrapped density function can be represented by the discrete Fourier transform

$$f(\theta) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \varphi_j \exp\{-\iota j\theta\}, \tag{1}$$

where $\iota = \sqrt{-1}$, and $\varphi_j = \varphi(j)$ is the jth Fourier coefficient obtained from $\varphi(t) = \mathrm{E}[\exp\{\iota t X\}]$. The equality in (1) is meant in the $\mathcal{L}^2$-sense and the series in the right-hand-side of (1) is known to converge to f: pointwise if f is differentiable, or uniformly if f is $\mathcal{C}^2$. In this article, our focus is on the class of wrapped α-stable distributions, which derive from the widely used α-stable distributions on $\mathbb{R}$ that possess many important properties. We denote an α-stable random variable by $S_\alpha(\tau, \beta, \mu)$ where $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\tau \in \mathbb{R}_+$ and $\mu \in \mathbb{R}$ are the indexes of stability, skewness, scale and shift, respectively. When $\beta = 0$, the subclass $S_\alpha(\tau, 0, \mu)$ is symmetric about μ. An α-stable random variable $S_\alpha(\tau, \beta, \mu)$ can be characterized by the fact that it has a domain of attraction, i.e., there exist a sequence $\{Y_n\}$ of independent and identically distributed (i.i.d.) random variables and sequences $\{a_n\}$ and $\{b_n\}$ of real positive numbers such that, as $n \to \infty$,

$$\frac{Y_1 + \ldots + Y_n}{b_n} + a_n \xrightarrow{\mathcal{D}} S_\alpha(\tau, \beta, \mu).$$

Hence, the α-stable class provides various limiting distributions, with the standard Central Limit Theorem as a special case (Y_i with finite variance and $\alpha = 2$). A property of an α-stable random variable is that, for $\alpha \in (0, 2)$, $\mathrm{E}[|S_\alpha(\tau, \beta, \mu)|^p] < \infty$ only when $p \in (0, \alpha)$, so that such random variables do not have a finite variance, except when $\alpha = 2$. For a complete presentation, see e.g. Feller (1971), or Samorodnitsky and Taqqu (1994).

The α-stable class is widely used, especially in finance (see e.g. Fama and Roll, 1968). Wrapped α-stable distributions are constructed via (1) by using the characteristic function of the α-stable of the real line, given by

$$\varphi(t) = \begin{cases} \exp\{-\tau^\alpha |t|^\alpha [1 - \iota\beta \operatorname{sgn} t \tan \frac{\alpha\pi}{2}] + \iota\mu t\}, & \text{if } \alpha \in (0,1) \cup (1,2], \\ \exp\{-\tau|t| + \iota\mu t\}, & \text{if } \alpha = 1. \end{cases}$$

The Fourier coefficients for a wrapped circular random variable correspond to the characteristic function at integer values for the unwrapped random variable (see e.g. Mardia, 1972, equation (3.4.25)). Thus, using (1), the density function of a wrapped α-stable random variable for $\theta \in [0, 2\pi)$, is given by

$$f(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{j=1}^{\infty} \exp\{-\tau^\alpha j^\alpha\} \cos\left\{ j(\theta - \mu) - \tau^\alpha j^\alpha \beta \tan\frac{\alpha\pi}{2}\right\}, \quad (2)$$

when $\alpha \in (0,1) \cup (1,2]$, and μ is conveniently redefined as $\mu \stackrel{\text{def}}{=} \mu \bmod(2\pi)$. Sometimes, τ is reparametrized by using the concentration parameter $\rho = \exp\{-\tau^\alpha\}$ (see JS, 2001 equation (2.2.18)). Two well known examples of α-stable random variables for which the density function admits a closed form expression are the normal with mean μ and variance $2\tau^2$, denoted as $S_2(\tau, 0, \mu)$ or as $N(\mu, 2\tau^2)$, and the Cauchy $S_1(\tau, 0, \mu)$, with density function $2\tau/(\pi[(x-\mu)^2 + 4\tau^2])$. The WN distribution corresponding to (2) with $\alpha = 2$ (β is then irrelevant) has the density given in (9). This is known to provide reasonable approximation and is similar in shape to the von Mises density (see e.g. JS, 2001, p. 45). The wrapped Cauchy distribution corresponds to the case $\alpha = 1$ and $\beta = 0$. The practical importance of this wrapped α-stable class can be explained as follows. From the fact that the Fourier coefficients of a wrapped circular model correspond to the characteristic function at integer values of the unwrapped model, one can directly deduce that many of the important properties of linear α-stables, as e.g. domain of attraction, closure under convolution, etc., are maintained after wrapping. Secondly, both the currently used von Mises distribution and the WN distribution do not provide sufficient degree of flexibility, within the class of symmetric distributions. We reproduce in Figure 1 some examples of WSαS densities for various choices of dispersion parameter τ and shape parameter α. Values of α smaller than two lead to densities with heavy tails which cannot be reproduced by WN densities with larger variance. In Figure 1, we plot over $[-\pi, \pi)$ three WSαS densities with $\alpha = 0.4, 0.8, 1.2$ and $\tau = 1$ (solid lines), and three WN densities ($\alpha = 2$) with $\tau = 0.5, 1, 1.2$ (dashed lines). All densities have $\mu = 0$. The various curves are easy to identify, since small values of α correspond to heavy tails, and large values of τ correspond to high dispersion. We can see that a WN density with the same extreme tail value as another WSαS density, would differ substantially in shape. To conclude, just as on the real line, WSαS densities allow for arbitrarily heavier tails than can both the WN density or the von Mises density, the latter also called circular normal and defined as

$$f(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp\{\kappa \cos(\theta - \mu)\}, \tag{3}$$

where I_n is the modified Bessel function of order n, and $\kappa > 0$ is the concentration parameter.

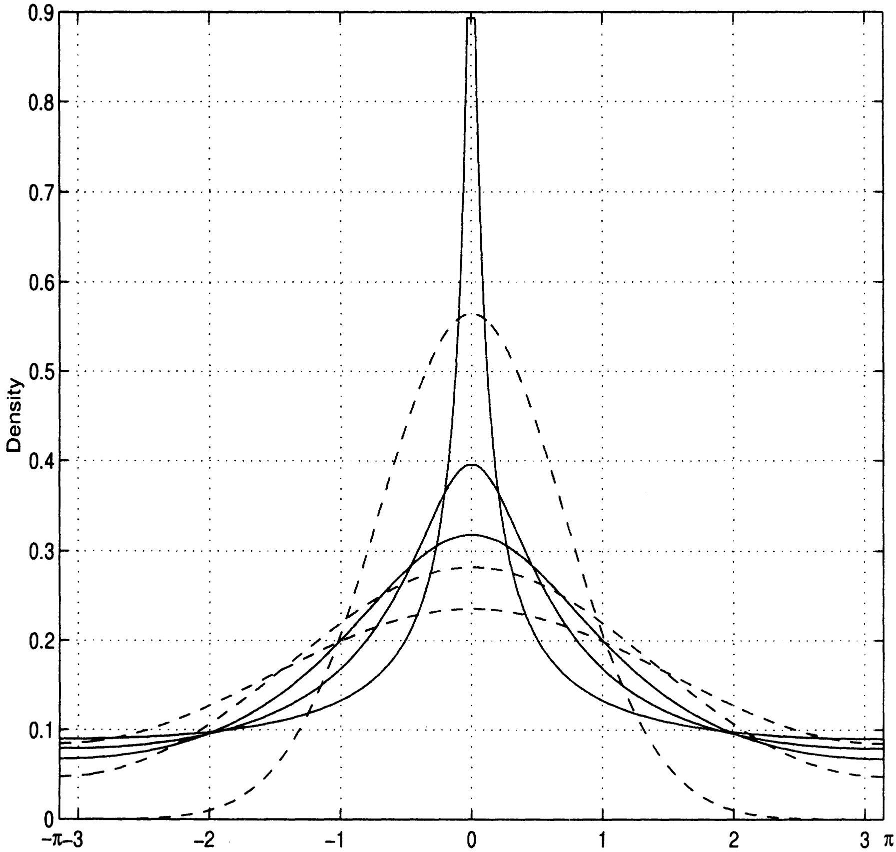

Figure 1. WSαS densities with $\alpha = 0.4$, 0.8, 1.2 and $\tau = 1$ (solid lines), and with $\alpha = 2$ and $\tau = 0.5$, 1, 1.2 (dashed lines); all with $\mu = 0$.

Clearly the WSαS class can provide better fit to data than the usual von Mises or WN distributions do and, as an illustrative example, we consider the cross-bedding azimuths data (with sample size 298) for the middle Kamthi formation described in Sengupta and Rao (1966, Table 1), where the von Mises model is not a suitable model. We computed the chi-square goodness-of-fit with the WSαS distribution with $\alpha = 1.6$, and with the von Mises distribution. The parameters of the WSαS distribution were estimated with the trigonometric method of moments estimator given in Section 3.1 (we obtain $\hat{\mu} = 0.016$ and $\hat{\tau} = -(\log \bar{R})^{1/\alpha} = 0.474$), and the parameters of the von Mises distribution were estimated by the maximum likelihood method (we obtain $\hat{\mu} = 0.016$ and $\hat{\kappa} = 2.260$). The P-value of the test for the WSαS distribution is 0.091, whereas the P-value for the von Mises distribution is 0.013, indicating a better fit for the WSαS class of distributions.

The organization of the rest of this article is as follows. Section 2 gives a saddlepoint approximation for the resultant length and provides some numerical results illustrating its accuracy under the uniform as well as under the WN distributions. These saddlepoint approximations provide very accurate approximations to P-values and to the power function of a test of uniformity against alternatives in the WSαS class. Section 3 introduces estimators for the parameters of WSαS distributions based on trigonometric moments, and proposes a method for testing hypothesis on the mean direction within the WSαS class using a conditional saddlepoint approximation. Section 4 provides a short discussion, and the important proofs can be found in the Appendix.

2. A Saddlepoint Approximation for the Distribution of the Resultant Length

In this section we mainly adapt the saddlepoint approximation to the problem of computing the distribution of the resultant length of n independent vectors in $\mathbb{R}^2$ when the distributions of their angles and of their lengths are given. A conditional saddlepoint approximation that can be used for inference on the mean direction is derived from this saddlepoint approximation in Section 3.2. The saddlepoint approximation was introduced into statistics by Daniels (1954), for deriving a very accurate approximation to the density function of the sample mean. Since then, it has been generalized to a variety of situations. It is known to provide approximations to small tail probabilities with high numerical accuracy, even with very small sample sizes. The exceptional tail behaviour of the saddlepoint approximation is also a consequence of its bounded relative error, at most of the order n^{-1}, and this uniformly over arbitrary compact sets. In contrast to this, the Edgeworth expansion is less accurate and it inherits the undesirable oscillations from its Hermite polynomials, sometimes leading to negative tail probabilities. Several extensions of Daniel's original formula have been proposed, see for example: Lugannani and Rice (1980) for tail probabilities, Field (1982) for M-estimators, Skovgaard (1987) for conditional distributions, Gatto and Ronchetti (1996) for marginal densities, etc. Some general texts or reviews are: Barndorff-Nielsen and Cox (1989), Field and Ronchetti (1990), and Field and Tingley (1997), the latter with emphasis on robust statistics.

In general, the resultant length R_n plays a central role in circular data analysis. For example, it can be seen as the total distance covered by a random walk where, after each step, the direction turns randomly through a specified angular distribution, and the length is either fixed or is governed by some other distribution. For the specific case where the vectors have unit length and their angular distribution is uniform, as in Pearson's random walk, the exact distribution of the resultant length R_n is given by the Kluyver integral, see Section 2.1. The distribution of R_n for other circular distributions is rather complex. Also, in view of the fact that the locally most powerful invariant test for testing H_0: $\rho = 0$, i.e. uniformity, against H_1: $\rho > 0$ within the WSαS class is of the form $R_n > c$ (see Theorem 6.6, p. 142, JS, 2001), the distribution of R_n under uniformity provides the critical values whereas the distribution under the alternatives can be used to find the power function of such a test. Proposition 2.1 below provides a saddlepoint approximation in a general setting which can be used to obtain the distributions under these WSαS alternatives. In Proposition 2.2 we provide the cumulant generating function which is the central component of the saddlepoint approximation. In Section 2.1, we provide numerical results of the saddlepoint approximation for the distribution of R_n when the data arise from a uniform distribution, i.e. under the null hypothesis above. Numerical results for the WN alternative are reported to Section 2.2.

PROPOSITION 2.1. *Suppose we have n independent random vectors in $\mathbb{R}^2$, with polar coordinates $(\theta_1, r_1), \ldots, (\theta_n, r_n)$, where $\theta_i \in [0, 2\pi)$ is the direction and $r_i > 0$ is the length, $i = 1, \ldots, n$. For $\lambda = (\lambda_1, \lambda_2)^{\mathrm{T}} \in \mathbb{R}^2$, we define*

$$K_n(\lambda) = \sum_{i=1}^{n} \log \left\{ \mathrm{E}[\exp\{r_i(\lambda_1 \cos\theta_i + \lambda_2 \sin\theta_i)\}] \right\} \tag{4}$$

as the sum of the cumulant generating functions of each sample value, in *terms of the Cartesian coordinates* $(r_i \cos\theta_i, r_i \sin\theta_i)$, $i = 1, \ldots, n$. *Let*

$$R_n = \left[\left(\sum_{i=1}^{n} r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} r_i \sin\theta_i\right)^2\right]^{\frac{1}{2}}$$

be the sample resultant length. If

$$h_n(\lambda) = [\det K_n''(\lambda)]^{\frac{1}{2}} \exp\{K_n(\lambda) - \lambda^{\mathrm{T}} K_n'(\lambda)\}, \tag{5}$$

$$\mathcal{D}(r) = \{\lambda \in \mathbb{R}^2 \mid \|K_n'(\lambda)\| < r\}, \text{ and } P_{\mathcal{D}n}(r) = \frac{1}{2\pi}\int_{\mathcal{D}(r)} h_n(\lambda)d\lambda,$$

then, for $r \geq 0$, *we have*

$$P[R_n < r] \;=\; P_{\mathcal{D}n}(r)\,[1 + O(n^{-1})], \tag{6}$$

as $n \to \infty$. *In the above expressions,* $K_n'(\lambda) = \partial/\partial\lambda\, K_n(\lambda)$, $K_n''(\lambda) = \partial^2/(\partial\lambda^{\mathrm{T}}\partial\lambda)K_n(\lambda)$, *and* $\|\cdot\|$ *denotes the Euclidean norm.*

For a proof of this, see the Appendix.

REMARK 2.1. From a practical point of view, it is important to note that the two-dimensional integral defining $P_{\mathcal{D}n}$ can be computed over the domain of the cumulant generating function, i.e. with respect to λ, the argument of the cumulant generating function, instead of t, which is a point in the domain of the sum of cosines and sum of sines (see also the Appendix). It is hence not necessary to solve a large number of saddlepoint equations $K_n'(\lambda) = t$ at each point t of the two-dimensional grid of integration (of the sum of cosines and sum of sines), as we would do if we were to integrate the joint saddlepoint approximation directly with respect to t. This makes $P_{\mathcal{D}n}$ useful for applications.

REMARK 2.2. It is easy to see that the domain of integration, characterized by $\|K_n'(\lambda)\| < r$, forms a simply connected set in the domain of the cumulant generating function. When $r < \infty$, $\mathcal{D}(r)$ is hence a compact set. This domain is not necessarily convex, and its central point is given by the stationary point of $K_n(\lambda)$, i.e. by $\lambda = (0, 0)^{\mathrm{T}}$.

REMARK 2.3. Given $(\theta_1, r_1), \ldots, (\theta_n, r_n)$ i.i.d., the bootstrap or nonparametric version of the saddlepoint approximation is obtained by using the empirical distribution of the sample in (4), yielding the bootstrap cumulant generating function

$$\hat{K}_n(\lambda) = -n \log n + n \log \left\{ \sum_{i=1}^{n} \exp\{r_i(\lambda_1 \cos\theta_i + \lambda_2 \sin\theta_i)\} \right\}.$$

The bootstrap saddlepoint approximation to the distribution of R_n is obtained by applying $\hat{K}_n$ in Proposition 2.1, and it has relative error $O_P(n^{-1})$ with respect to the bootstrap distribution (that would be obtained with a very large number of resamplings), see Wang (1990) for justifications.

The next proposition gives the form of the cumulant generating function for a WSαS underlying distribution, which allows to compute the saddlepoint approximation of Proposition 2.1 in this case.

PROPOSITION 2.2. *Given a random angle $\theta \in [0, 2\pi)$ with a WSαS distribution, the cumulant generating function of $(\cos\theta, \sin\theta)^T$ is given by*

$$K(\lambda) = \log \left\{ I_0(||\lambda||) + 2\sum_{j=1}^{\infty} \exp\{-\tau^\alpha j^\alpha\} \cos\{j[\mu - \arg\{\lambda_1 + \iota\lambda_2\}]\} I_j(||\lambda||) \right\}, \tag{7}$$

where I_j is the modified Bessel function of the first kind of integer order j.

For a proof of this, see the Appendix.

With K given by (7), taking $K_n = nK$ in Proposition 2.1 gives the saddlepoint approximation for n i.i.d. unit vectors with WSαS angles.

2.1 *Uniform null distribution.* In the following example we consider the case where the angles are independent with uniform distribution and where the associated vectors have unit length. As previously written, the saddlepoint approximation to the resultant length can be used for obtaining the critical values of the test H_0: $\rho = 0$, against H_1: $\rho > 0$, within the WSαS class. This approximation could however also be settled in the context of the classical Pearson's random walk problem.

EXAMPLE 2.1. Suppose we have n unit length independent vectors in $\mathbb{R}^2$, with angles $\theta_1, \ldots, \theta_n$ uniformly distributed over $[0, 2\pi)$. The cumulant generating function (4) for this case is the limiting value of (7) as τ tends to infinity,

$$K_n(\lambda) = n \log\{I_0(||\lambda||)\}. \tag{8}$$

For this simple case, it is known (see e.g. Mardia, 1972, p. 94) that the exact distribution of R_n can be obtained by Kluyver's integral, i.e. by

$\mathrm{P}[R_n < r] = r\int_0^\infty J_0^n(v)J_1(rv)dv$, J_n denoting the Bessel function of the first kind of integer order n. Moreover, an asymptotic approximation to this probability (provided by Rayleigh) is given by $1 - \exp\{-r^2/2\}$. In Table 1 we have the cumulative distributions of the resultant length, computed by: numerical integration (values tabulated by Greenwood and Durand, 1954), by our saddlepoint approximation (6), and by the Rayleigh's approximation above. These distributions are respectively denoted by: $\mathrm{P_E}$, $\mathrm{P_S}$, and $\mathrm{P_R}$, and we refer to $\mathrm{P_E}$ as the "exact" distribution. Table 1 shows that with the sample size $n = 7$ the saddlepoint approximation performs very well in the tails of the distribution, reflecting the fact that it has a bounded relative error. Table 1 also indicates that for $n = 7$, the critical region of the test of uniformity against WSαS alternatives with size $\alpha = 0.05$ is approximately $[4.5, 7]$, according to both the exact distribution and the saddlepoint approximation.

TABLE 1. RESULTANT LENGTH UNDER UNIFORMITY OF ANGLES AND FIXED INDIVIDUAL LENGTHS: EXACT DISTRIBUTION ($\mathrm{P_E}$), SADDLEPOINT ($\mathrm{P_S}$), AND RAYLEIGH ($\mathrm{P_R}$) APPROXIMATIONS; $n = 7$. THE EXACT PROBABILITIES WERE OBTAINED BY NUMERICAL INTEGRATION.

r	$\mathrm{P_E}[R_n < r]$	$\mathrm{P_S}[R_n < r]$	$\mathrm{P_R}[R_n < r]$
0.5	0.032	0.031	0.035
1.0	0.125	0.124	0.133
1.5	0.261	0.258	0.275
2.0	0.418	0.413	0.435
2.5	0.575	0.572	0.591
3.0	0.714	0.709	0.724
3.5	0.823	0.821	0.826
4.0	0.900	0.899	0.898
4.5	0.951	0.951	0.945
5.0	0.979	0.980	0.972
5.5	0.992	0.995	0.987
6.0	0.998	1.000	0.994

REMARK 2.4. Note that, because of the U-statistic representation of the squared resultant length, $R_n^2 = n + \sum\sum_{i\neq j}\cos(\theta_i - \theta_j)$, the distribution of R_n could also be approximated by the general saddlepoint approximation given by Gatto and Ronchetti (1996).

REMARK 2.5. Under uniformity of the angles θ_i, the general case with n independent random vectors in $\mathbb{R}^2$ with polar coordinates (θ_i, r_i) with θ_i and r_i independent, $i = 1, \ldots, n$, could be simplified as follows. By considering the expansion of the function I_0, it can easily be seen that

$$K_n(\lambda) = \sum_{i=1}^{n} \log\left\{1 + \sum_{j=1}^{\infty} c_j(\lambda)\mu_{2j}^{(i)}\right\},$$

where $\mu_j^{(i)} = \int r^j dF_i(r)$, F_i representing the distribution of the ith length r_i, and $c_j(\lambda) = (||\lambda||^{2j}/[2^j j!])^2$ (e.g. $c_1(\lambda) = ||\lambda||^2/4$, $c_2(\lambda) = ||\lambda||^4/64$, $c_3(\lambda) = ||\lambda||^6/2304$, etc.). This K_n can be directly inserted in the saddlepoint approximation (6).

2.2 *WN alternative distribution.* Wrapping the normal random variable with mean μ and variance σ^2 around the unit circle leads to the WN random variable, namely $WN(\mu,\sigma^2) \stackrel{\text{def}}{=} S_2(\sigma/\sqrt{2}, 0, \mu) \bmod(2\pi)$. According to (2), its density function is given by

$$f(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{j=1}^{\infty} \exp\{-\frac{1}{2}j^2\sigma^2\}\cos(j[\theta - \mu]), \quad \theta \in [0, 2\pi). \tag{9}$$

This can be seen as the particular alternative H_1: $\rho = \exp\{-\sigma^2/2\} > 0$ with $\alpha = 2$, in the context of the locally most powerful invariant test of uniformity, and the saddlepoint approximation provides the value of the power function at such alternative. The following example shows the effectiveness and the numerical precision of the saddlepoint approximation.

EXAMPLE 2.2. We consider a sample of size $n = 6$ independent unit vectors with angles from the WN distribution, with $\mu = 0$ and $\sigma^2 = 4$. The cumulant generating function for this particular case is directly obtained from (7), yielding

$$K_n(\lambda) = n\log\left\{I_0(||\lambda||) + 2\sum_{j=1}^{\infty}\exp\left\{-\frac{j^2\sigma^2}{2}\right\}\cos\{j[\arg\{\lambda_1 + \iota\lambda_2\} - \mu]\}I_j(||\lambda||)\right\}. \tag{10}$$

The first and the second order derivatives of K_n, necessary for the computation of the saddlepoint approximation can be obtained with the help of automatic symbolic computation (Maple). The infinite sum in (10) converges very fast, so that only the first few summations are necessary for numerical evaluations. Table 2 gives the saddlepoint approximation of the distribution of R_n, after renormalization. The distributions appearing in Table 2 are: $\mathrm{P_E}$, the "exact" distribution based on 10^6 Monte Carlo samplings, and $\mathrm{P_S}$, the saddlepoint approximation. As in the previous example, we can see that the saddlepoint approximation is extremely accurate. We next give the absolute and the absolute relative errors, respectively given by $|\mathrm{P_E} - \mathrm{P_S}|$ and $|\mathrm{P_E} - \mathrm{P_S}|/\min\{\mathrm{P_E}, (1 - \mathrm{P_E})\}$. Figure 2 shows that the absolute error is generally very small, and that the absolute relative error remains generally bounded below 10%, if we except one small peak at about 0.02 due to the error in the numerical integration ($\mathcal{D}(r)$ is discretized by small squares and h_n is evaluated at the centers of these squares only), and another peak as we approach 1, which is due to the limiting small denominator in the computation of the absolute relative error (see Table 2).

TABLE 2. RESULTANT LENGTH UNDER WN ANGLES AND FIXED INDIVIDUAL LENGTHS: EXACT DISTRIBUTION ($\mathrm{P_E}$) AND SADDLEPOINT APPROXIMATION ($\mathrm{P_S}$); $\mu = 0$, $\sigma^2 = 4$, $n = 6$. THE EXACT PROBABILITIES WERE OBTAINED BY 10^6 MONTE CARLO SIMULATIONS.

r	$\mathrm{P_E}[R_n < r]$	$\mathrm{P_S}[R_n < r]$
0.5	0.036	0.035
1.0	0.129	0.126
1.5	0.266	0.267
2.0	0.434	0.424
2.5	0.595	0.584
3.0	0.734	0.725
3.5	0.844	0.836
4.0	0.921	0.915
4.5	0.965	0.963
5.0	0.988	0.988
5.5	0.998	0.999
6.0	1.000	1.000

3. Inference on Parameters of WSαS Distribution

3.1 *Trigonometric method of moments estimators.* One can see from (2) that reductions through sufficiency and invariance are generally not available for WSαS distributions (especially for inference on the mean direction), and the maximum likelihood approach, which would remain purely numerical, can hardly yield useful theoretical insights. We propose a class of estimators obtained by extending the method of moments estimation of the Cartesian coordinates case. In this case, it is often convenient to reparametrize in terms of the concentration $\rho = \exp\{-\tau^\alpha\}$. For $j = 1, 2, \ldots$, the jth trigonometric moment of any circular random variable θ is given by

$$\mathrm{E}[e^{\iota j\theta}] = \mathrm{E}[\cos(j\theta) + \iota\sin(j\theta)] = \rho_j e^{\iota\mu_j}, \tag{11}$$

the last expression is in polar coordinates so that $\rho_j = |\mathrm{E}[\cos(j\theta)] + \iota\mathrm{E}[\sin(j\theta)]|$ and $\mu_j = \arg\{\mathrm{E}[\cos(j\theta)] + \iota\mathrm{E}[\sin(j\theta)]\}$. From the fact that the Fourier coefficients of a wrapped random variable are the characteristic function at integer

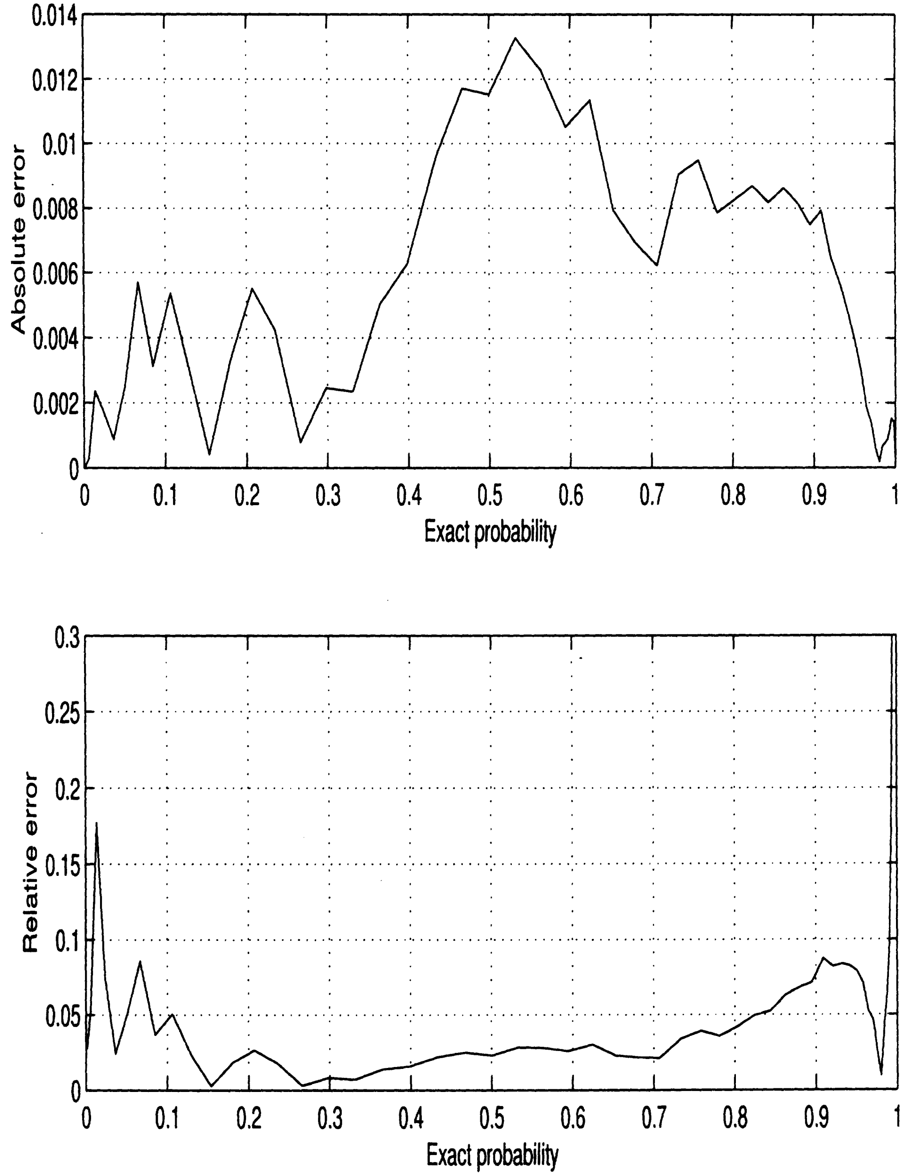

Figure 2. Saddlepoint approximation for the distribution of the resultant length under the WN distribution, $\mu = 0$, $\sigma^2 = 4$, $n = 6$. Upper figure: absolute error $|\mathrm{P_E} - \mathrm{P_S}|$. Lower figure: absolute relative error $|\mathrm{P_E} - \mathrm{P_S}| / \min\{\mathrm{P_E}, (1 - \mathrm{P_E})\}$. $\mathrm{P_E}$: exact distribution obtained by 10^6 simulations. $\mathrm{P_S}$: saddlepoint approximation to the distribution. The abscissae are the exact probabilities.

values of the corresponding unwrapped random variable, we have for the WSαS case

$$\mathrm{E}[e^{\iota j\theta}] = \rho^{|j|^\alpha} e^{\iota\mu j},$$

so that, together with (11),

$$\rho_j = \rho^{j^\alpha} \text{ and } \mu_j = j\mu.$$

Replacing these ρ_j and μ_j by their sample versions, for $j = 1$, we obtain

$$\hat{\rho} = n^{-1}R_n, \text{ and } \hat{\mu} = \arg\left\{\sum_{i=1}^{n} \cos\theta_i + \iota\sum_{i=1}^{n} \sin\theta_i\right\},$$

which will be referred to as the trigonometric method of moments estimators (TMME) of ρ and μ, based on the first trigonometric moment.

EXAMPLE 3.1. In this example we derive the TMME of $\sigma^2 = 2\tau^2$ for the WN distribution, its influence function, as well as the saddlepoint approximation to its distribution, which is a simple transform of the one for the resultant length given by (6). Equating $e^{-\sigma^2/2}$ to $\bar{R} = n^{-1}R_n$, the mean resultant length, and solving for σ^2, gives the TMME of σ^2,

$$\hat{\sigma}^2 = -2\log\{\bar{R}\}. \tag{12}$$

Let F denote the underlying WN distribution, and let $\hat{F}$ denote the empirical distribution of a sample drawn from F. Clearly $\hat{\sigma}^2$ admits a functional representation denoted $T(\hat{F})$. From there, the influence function of T at F, at a point $\theta \in [0, 2\pi)$ can be easily computed with

$$IF(\theta; T, F) = \frac{\partial}{\partial\varepsilon}T((1-\varepsilon)F + \varepsilon\Delta_\theta)|_{\varepsilon=0} = 2[1 - \exp(\sigma^2/2)\cos(\theta - \mu)],$$

where Δ_θ is the Dirac distribution with mass one at θ. The influence function measures the standardized effect on the estimator resulting from a contamination of F at point θ. This influence function is plotted in Figure 3, in both Cartesian and polar coordinates, for the WN case with $\mu = 0$ and $\sigma^2 = 4$. In these plots, we can also see that the gross-error sensitivity, defined by $\gamma(T, F) = \sup_\theta|IF(\theta; T, F)|$, is finite ($\gamma = 16.778$). Hence, $\hat{\sigma}^2$ is bias-robust, since the gross-error sensitivity measures the largest bias that a contamination can induce to the estimator. The smooth behaviour of the influence function shows also that the estimator would slowly fluctuate after small fluctuations in the observations. This is formally measured by the local-shift sensitivity $\sup_{\theta_1\neq\theta_2}|IF(\theta_1; T, F) - IF(\theta_2; T, F)|/|\theta_1 - \theta_2|$.

The influence function was originally proposed by Hampel (1974). The saddlepoint approximation to the distribution of $\hat{\sigma}^2$ in the WN model is based on (6) because of the relation $\mathrm{P}[\hat{\sigma}^2 < s] = 1 - \mathrm{P}[R_n < ne^{-s/2}]$. It is hence given by $1 - P_{\mathcal{D}n}(ne^{-s/2})$, and the saddlepoint approximation to the density would be $p_{\mathcal{D}n}(ne^{-s/2})n\ e^{-s/2}/2$, where $p_{\mathcal{D}n}(s) = d/dsP_{\mathcal{D}n}(s)$. Figure 4 gives this approximation for the distribution of $\hat{\sigma}^2$, for $n = 6$ $WN(0,4)$ independent observations, as well as the exact distribution based on 10^6 simulations. In order to emphasize the tail behaviour of the approximation, the probabilities $\mathrm{P}[\hat{\sigma}^2 < s]$ are in the logit scale, that is the scale on left axis of Figure 4 is $\log\{P/(1-P)\}$, and the scale on the right axis is P, where P denotes both exact and saddlepoint probabilities. As we can see, the saddlepoint approximation behaves very well.

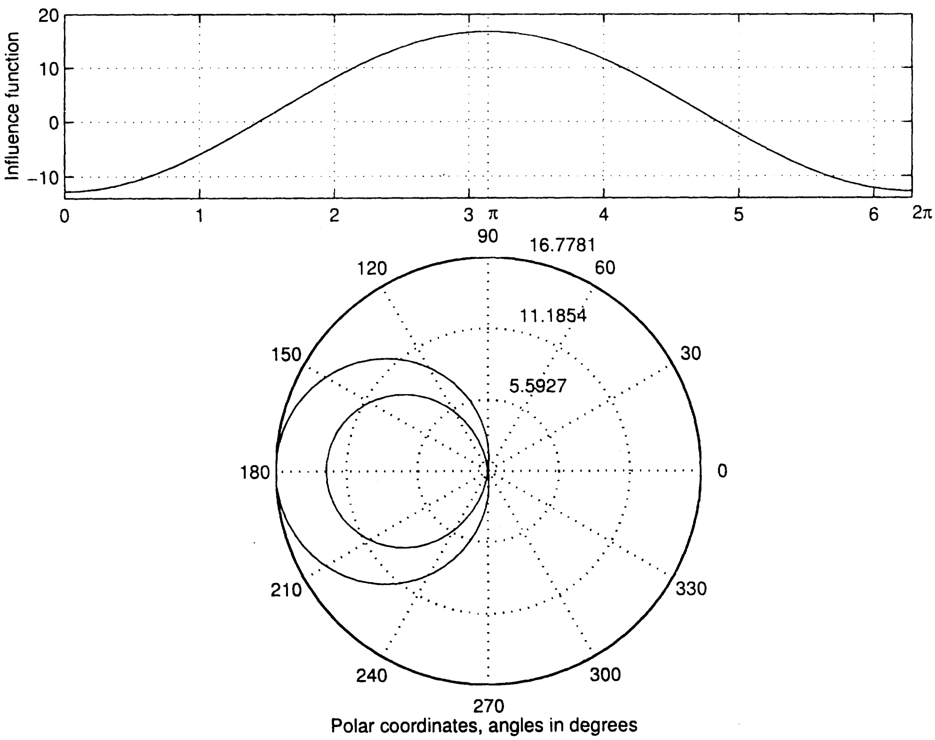

Figure 3. Influence function of the TMME of the variance under the WN distribution, $\mu = 0$, $\sigma^2 = 4$. The upper plot is in Cartesian coordinates, and the lower plot in polar coordinates.

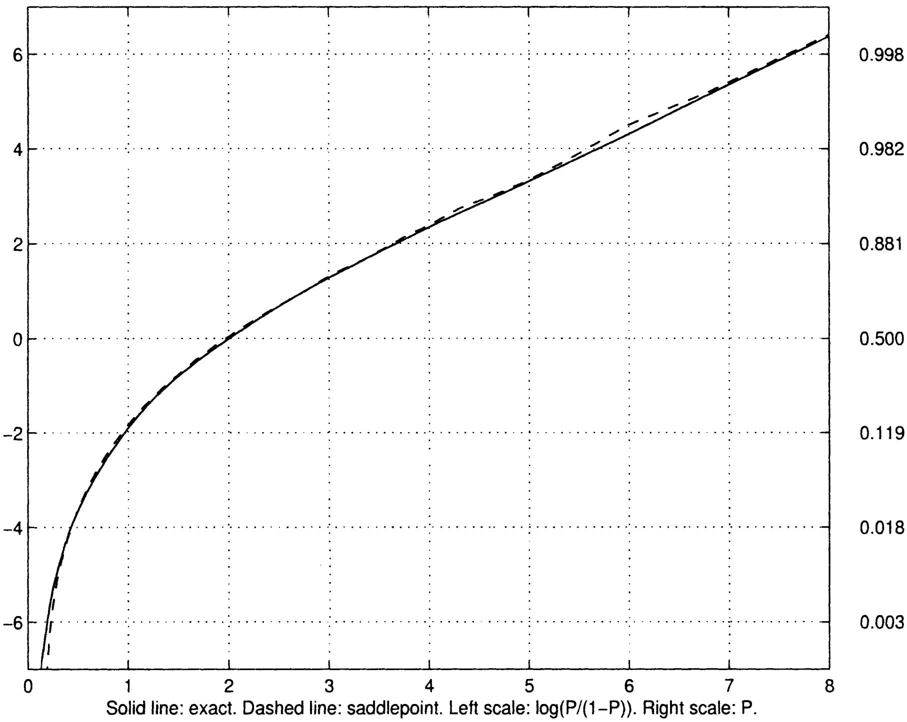

Figure 4. Saddlepoint approximation for the distribution of the TMME of the variance under the WN distribution, $\mu = 0$, $\sigma^2 = 4$, $n = 6$. The scale on left axis is $\log\{P/(1-P)\}$, and the scale on the right axis is P, P denoting both exact and saddlepoint lower probabilities. The exact probabilities are obtained by 10^6 simulations. Solid line: exact. Dashed line: saddlepoint.

REMARK 3.1. This TMME can be extended by using the second trigonometric moment to the estimation of the stability index α. This estimator admits a simple representation, even though the estimation in the (non-wrapped) symmetric α-stable models is complicated (see e.g. Fama and Roll, 1971).

3.2 *Inference on the mean direction.* In this section we discuss the problem of inference on the mean direction μ of a WSαS model, when the scale parameter τ or the concentration parameter ρ are unknown. We propose a procedure for testing hypotheses which exploits both a conditional saddlepoint approximation and the trigonometric method of moments estimators introduced in Section 3.1. In this procedure, the nuisance parameter ρ, equivalently τ, is eliminated by conditioning on the resultant length. In a first step, Proposition 3.1 gives the asymptotic normal distribution of the tangent of the mean direction, and shows that this distribution depends on the nuisance parameter ρ. In this context, Gatto (2000) proposes a simultaneous test on the mean direction and the scale parameter with a test statistic based on the exponent of the saddlepoint approximation of the density of M-estimators, which is asymptotically chi-squared distributed, up to the second order.

PROPOSITION 3.1. *Suppose $\theta_1, \ldots, \theta_n$ are i.i.d. WSαS random angles in $[0, 2\pi)$, i.e. with trigonometric moments given by $E[e^{\iota j\theta}] = \rho^{|j|^\alpha} e^{\iota\mu j}$, $j = 1, 2, \ldots$. Let $\hat{\mu} = \arg\{\sum_{i=1}^n \cos\theta_i + \iota\sum_{i=1}^n \sin\theta_i\}$ be the associated TMME of μ associated with these random angles. Then the limiting distribution of the tangent of the mean direction is given by*

$$\sqrt{n}(\tan\hat{\mu} - \tan\mu) \xrightarrow{\mathcal{D}} N(0, \zeta^2), \tag{13}$$

where

$$\zeta^2 = \frac{1 - \rho^{2^\alpha}}{2\rho^2\cos^4\mu}$$

is the asymptotic variance.

For a proof, see the Appendix.

PROPOSITION 3.2. *Suppose $\theta_1, \ldots, \theta_n$ are i.i.d. random angles in $[0, 2\pi)$ with mean direction $\mu \in [0, 2\pi)$. For $\lambda = (\lambda_1, \lambda_2)^{\mathrm{T}} \in \mathbb{R}^2$, we define $K_n(\lambda)$ as in (4) and $h_n(\lambda)$ as in (5). Then, for $r \geq 0$, $\theta \in [0, 2\pi)$,*

$$\begin{aligned} \mathcal{C}(r) &= \{\lambda \in \mathbb{R}^2 \mid \|K_n'(\lambda)\| = r\}, \\ \mathcal{A}(\theta) &= \{\lambda \in \mathbb{R}^2 \mid 0 \leq \arg\{K_{n1}'(\lambda) + \iota K_{n2}'(\lambda)\} < \theta\}, \end{aligned}$$

arg *defined in $[0, 2\pi)$, and for*

$$P_{\mathcal{A}\mathcal{C}n}(\theta \mid r) = \frac{\int_{\mathcal{C}(r)\cap\mathcal{A}(\theta)} h_n(\lambda)d\lambda}{\int_{\mathcal{C}(r)} h_n(\lambda)d\lambda}, \tag{14}$$

we have,

$$P[0 \leq \hat{\mu} < \theta \mid R_n = r] = P_{\mathcal{A}\mathcal{C}n}(\theta \mid r)\,[1 + O(n^{-1})], \tag{15}$$

as $n \to \infty$, where $K_{nj}'(\lambda) = \partial/\partial\lambda_j K_n(\lambda_1, \lambda_2)$, $j = 1, 2$.

For a proof, see the Appendix.

REMARK 3.2. We can note that $\mathcal{C}(r)$ is the circle of radius r in the sum of cosines and sum of sines domain, whereas $\mathcal{C}(r) \cap \mathcal{A}(\theta)$ is the arc of the same circle corresponding to all angles in $[0, \theta)$. In the cumulant generating function domain, these correspond to a closed path and to a portion of this path, respectively. Integrating in this latter domain is convenient, as explained by Remark 2.1.

REMARK 3.2. By (5), $1/(2\pi)h_n(\lambda)$ gives us the saddlepoint approximation to the joint density of $(\sum_{i=1}^n r_i \cos\theta_i, \sum_{i=1}^n r_i \sin\theta_i)$ at $t = K_n'(\lambda)$. By a simple change of variables we could obtain an approximation to the joint density of $(\hat{\mu}, R_n)$ at $t = (\theta, r)$, which would be given by $r/(2\pi)h_n(\lambda)$, where $K_n'(\lambda) = (r\cos\theta, r\sin\theta)$.

Suppose we are interested in testing the hypothesis H_0: $\mu = \mu_0$, for a specified $\mu_0 \in [0, 2\pi)$, against the general alternative H_1: $\mu \neq \mu_0$, and with size $\varepsilon \in (0,1)$. Since μ is a location parameter, $\hat{\mu}$ is location equivariant, and R_n is location invariant, we can rather refer to the shifted sample $\theta_i' = \theta_i - \mu_0$, to the associated mean direction $\hat{\mu}' = \hat{\mu} - \mu_0$, and to the associated null hypothesis H_0: $\mu' = 0$. From Proposition 3.1, the expression for the asymptotic variance under H_0 would then become

$$\zeta^2 = \frac{1-\rho^{2^\alpha}}{2\rho^2}.$$

From $\bar{R} \xrightarrow{\mathcal{P}} \rho$, Slutsky's theorem, Proposition 3.1 with the shifted sample, and an application of the delta-method (noting that $\arctan'(0) = 1$), we can assert that under H_0, $\sqrt{n}\hat{\mu}'$ is approximately distributed as $N(0, \hat{\zeta}^2)$, where

$$\hat{\zeta}^2 = \frac{1-\bar{R}^{2^\alpha}}{2\bar{R}^2}.$$

The fact that $\tan(\theta) = \tan(\theta + \pi)$ is not a real drawback in showing this last equivalence, since a preliminary rough inspection of the data allows to distinguish the pole from the anti-pole. It follows that the asymptotic distribution of the conditional statistic $\sqrt{n}\hat{\mu}' \mid \bar{R}$ does not involve the nuisance parameter ρ, so that this conditional statistic is asymptotically pivotal. It can hence be used for testing an hypothesis on the mean direction. Since the saddlepoint approximation under H_0 of $(\hat{\mu} - \mu_0) \mid R_n$, which is substantially

$\sqrt{n}\hat{\mu}' \mid \bar{R}$, is given by Proposition 3.2, a rejection region of size ε for this test based on the conditional saddlepoint approximation (15), would be given by

$$\{(\theta_1, \ldots, \theta_n) \in [0, 2\pi)^n \mid 0 \leq \hat{\mu} - \mu_0 < \hat{q}_1 \text{ or } \hat{q}_2 \leq \hat{\mu} - \mu_0 < 2\pi\},$$

where the quantiles $\hat{q}_1$ and $\hat{q}_2$ are solutions of $P_{\mathcal{A}Cn}(\hat{q}_1 \mid n\bar{R}^{\text{obs}}) = \varepsilon/2$ and $P_{\mathcal{A}Cn}(\hat{q}_2 \mid n\bar{R}^{\text{obs}}) = 1 - \varepsilon/2$, and $\bar{R}^{\text{obs}}$ is a realization of $\bar{R}$. Here, the saddlepoint approximation $P_{\mathcal{A}Cn}$ is similar to the one given by (14) except that the parameter ρ is replaced by $\bar{R}^{\text{obs}}$ in the formula. This substitution is possible as long as second order accuracy is desired. Given $\bar{R} = \bar{R}^{\text{obs}}$, the quantity $\hat{\mu}'/\hat{\zeta}$ admits the second order Edgeworth expansion of the form

$$\Phi(\theta) + n^{-\frac{1}{2}} p(\theta; \rho, \bar{R}^{\text{obs}}) \phi(\theta) + \mathrm{O}(n^{-1}), \tag{16}$$

where Φ and ϕ are the standard normal distribution and its density, and $p(\theta; \rho, \bar{R}^{\text{obs}})$ is a polynomial of degree two in θ. From $\bar{R} = \rho + \mathrm{O}_{\mathrm{P}}(n^{-1/2})$, it follows that replacing ρ by $\bar{R}^{\text{obs}}$ in (16) does not increase the asymptotic error, meaning that second order accuracy is maintained. Hence, ρ can similarly be replaced by $\bar{R}^{\text{obs}}$ in the conditional saddlepoint approximation (14), if we consider it as a second order approximation to the Edgeworth expansion (16). A confidence interval for μ with coverage rate $1 - \varepsilon$ could be defined as the set of values $\mu_0 \in [0, 2\pi)$ for which the sample would be outside the rejection region above.

REMARK 3.4. It is shown that applying this same conditioning to the mean direction in the von Mises model, determined by (3), would totally eliminate the dependence on the nuisance parameter $\kappa > 0$. Indeed, under the von Mises model, the conditional distribution of the mean direction given the resultant length does not depend on the concentration parameter κ at all (not only asymptotically).

REMARK 3.5. In the bootstrap case, the saddlepoint approximation can also be used to obtain second order accurate confidence intervals in conjunction with the bias-corrected accelerated (BC_a) method by Efron (1987). The saddlepoint approximation to $\mathrm{P}[0 \leq \hat{\mu} < \theta]$ can be obtained by integrating the joint saddlepoint approximation of the density of the sum of cosines and sum of sines over an appropriate slice of its domain. More precisely, we would need to consider the bootstrap version of $P_{\mathcal{A}n}(\theta) = \int_{\mathcal{A}(\theta)} h_n(\lambda) d\lambda / (2\pi)$, which we denote by $\hat{P}_{\mathcal{A}n}(\theta)$. It can be obtained by following the lines in Remark 2.3. It can be shown that the influence function of $\hat{\mu}$ at point θ, for the WSαS model, is given by $\sin(\mu-\theta)/\rho$. Hence, the acceleration constant of the BC_a method admits the simple form

$$\hat{a} = \frac{\sum_{i=1}^{n}\sin^3(\hat{\mu}-\theta_i)}{6[\sum_{i=1}^{n}\sin^2(\hat{\mu}-\theta_i)]^{\frac{3}{2}}};$$

compare with the general equation (7.3) in Efron (1987). The bias-correction is given by $\hat{z}_0 = \Phi^{(-1)}\{\hat{P}_{An}(\hat{\mu})\}$, where $\Phi^{(-1)}$ is the inverse of the standard normal distribution, see equation (4.1) in Efron (1987). Hence, the BC_a confidence interval with coverage rate $1-2\varepsilon$ is given by

$$\left(\hat{P}_{An}^{(-1)}\left\{\Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_\varepsilon}{1-\hat{a}(\hat{z}_0+z_\varepsilon)}\right)\right\}, \hat{P}_{An}^{(-1)}\left\{\Phi\left(\hat{z}_0 + \frac{\hat{z}_0 - z_{1-\varepsilon}}{1-\hat{a}(\hat{z}_0-z_{1-\varepsilon})}\right)\right\}\right),$$

where $\Phi(z_\varepsilon)=\varepsilon$, and where $\hat{P}_{An}^{(-1)}$ is the quantile function of $\hat{P}_{An}$, see formula (3.8) in Efron (1987). In the bootstrap case we can thus avoid the use of a pivotal statistic.

The numerical accuracy of the conditional saddlepoint approximation (15) is shown by the following example, for the WN model.

EXAMPLE 3.2. In this example we are interested in the distribution of the mean direction of $n=6$ i.i.d. unit vectors, conditioned to have resultant length R_n of 1. The distribution of the random angles of the unit vectors is the WN with mean zero and variance $-2\log(1/n) \simeq 3.584$. We approximate the conditional distribution of the mean direction by using the saddlepoint approximation provided by Proposition 3.2, with a final renormalization. We compute the "exact" distribution based on 95637 retained samples from a total number of $2\cdot 10^6$ simulations. In the conditional simulation, we first generate $n-1$ random angles from the WN distribution. If with the unit vectors associated to these $n-1$ angles it is possible, by adding a last unit vector, to obtain a total resultant length of 1, then we retain the first $n-1$ angles, otherwise we reject them. Once we accept a sample of $n-1$ angles, there remain only two possible values for the missing nth angle, and one of these two values is randomly selected, according to a probability that depends on the underlying WN distribution. The numerical results are shown in Table 3, where the exact distribution is denoted by $\mathrm{P_E}$, the saddlepoint approximation by $\mathrm{P_S}$, the absolute error and the relative absolute error of the saddlepoint approximation by $AE = \mid \mathrm{P_E}-\mathrm{P_S}\mid$ and $RAE = \mid \mathrm{P_E}-\mathrm{P_S}\mid / \min\{\mathrm{P_E}, 1-\mathrm{P_E}\}$, respectively. As we can see, the saddlepoint approximation leads to high accuracy over the entire range of the mean direction.

TABLE 3. DISTRIBUTION OF THE MEAN DIRECTION CONDITIONED TO HAVE A RESULTANT LENGTH 1, OF $n = 6$ WN ANGLES WITH MEAN 0 AND VARIANCE $-2\log(1/6) \simeq 3.584$: EXACT DISTRIBUTION ($\mathrm{P_E}$), SADDLEPOINT APPROXIMATION ($\mathrm{P_S}$), ABSOLUTE ERROR ($AE = |\mathrm{P_E} - \mathrm{P_S}|$), RELATIVE ABSOLUTE ERROR ($RAE = |\mathrm{P_E} - \mathrm{P_S}| / \min\{\mathrm{P_E}, 1 - \mathrm{P_E}\}$). THE EXACT PROBABILITIES WERE OBTAINED BY 95637 RETAINED VALUES FROM $2 \cdot 10^6$ SIMULATIONS.

θ	$\mathrm{P_E}[\hat{\mu} < \theta]$	$\mathrm{P_S}[\hat{\mu} < \theta]$	$AE(\theta)$	$RAE(\theta)$
0.157	0.039	0.038	0.001	0.029
0.314	0.077	0.075	0.002	0.024
0.471	0.113	0.105	0.008	0.072
0.628	0.150	0.129	0.021	0.137
0.785	0.184	0.164	0.020	0.111
1.100	0.248	0.229	0.018	0.074
1.414	0.302	0.279	0.023	0.078
1.728	0.349	0.324	0.026	0.073
2.042	0.390	0.368	0.021	0.055
2.356	0.425	0.408	0.017	0.039
2.670	0.457	0.449	0.008	0.018
2.985	0.487	0.488	0.001	0.001
3.299	0.516	0.519	0.003	0.006
3.613	0.546	0.557	0.011	0.025
3.927	0.578	0.598	0.020	0.048
4.241	0.612	0.638	0.026	0.067
4.555	0.652	0.683	0.030	0.087
4.869	0.698	0.728	0.029	0.097
5.184	0.752	0.777	0.025	0.100
5.498	0.813	0.842	0.029	0.157
5.655	0.848	0.877	0.029	0.189
5.812	0.886	0.901	0.015	0.133
5.969	0.923	0.932	0.008	0.106
6.126	0.962	0.969	0.007	0.186
6.283	1.000	1.000	0.000	0.000

4. Conclusion

In this article, we motivate the WSαS class for circular models and we treat some related inferential problems within this class, by using the TMME and the saddlepoint approximation. In all our numerical examples we considered small sample sizes and the saddlepoint approximations yield very accurate results. As in other situations, the saddlepoint approximation is a useful and accurate technique, especially in presence small or moderate sample sizes, or when extreme tail probabilities are to be approximated.

Appendix

PROOF OF PROPOSITION 2.1. Let us denote by $f_{C,S}$ the joint density of $(\sum_{i=1}^{n}\cos\theta_i, \sum_{i=1}^{n}\sin\theta_i)^{\mathrm{T}}$. The joint saddlepoint approximation to $f_{C,S}$ at $t \in \mathbb{R}^2$ is given by,

$$g_{C,S}(t) = \frac{1}{2\pi}[\det K_n''(\lambda)]^{-\frac{1}{2}} \exp\{K_n(\lambda) - \lambda^{\mathrm{T}}t\},$$

where $K_n(\lambda)$ is the joint cumulant generating function given by (4), and $\lambda \in \mathbb{R}^2$ is the saddlepoint defined by the equation $K_n'(\lambda) = t$. It follows that $f_{C,S}(t) = g_{C,S}(t)[1+\mathrm{O}(n^{-1})]$, and the error term holds uniformly for all t in a compact region, see Field (1982) p. 677. A saddlepoint approximation to the distribution of the resultant length R_n can be obtained by integrating $g_{C,S}(t)dt$ over the compact region $\{t = (t_1, t_2)^{\mathrm{T}} \in \mathbb{R}^2 \mid t_1^2 + t_2^2 < r^2\}$. Furthermore, the change of variable of integration $t \to \lambda$ (see Remark 2.1) involves the Jacobian $\det K''(\lambda)$, and the integral of $\det K''(\lambda) g_{C,S}(K'(\lambda))d\lambda$ over the compact region $\mathcal{D}(r)$ (see Remark 2.2) leads to the formula for $P_{\mathcal{D}n}$ of Proposition 2.1 and to (6). □

PROOF OF PROPOSITION 2.2. The joint moment generating function at $\lambda = (\lambda_1, \lambda_2)^{\mathrm{T}} \in \mathbb{R}^2$ of the cosine and the sine of a WSαS angle is given by

$$M(\lambda) = \int_0^{2\pi} \exp\{\lambda_1\cos\theta + \lambda_2\sin\theta\}\Big\{\frac{1}{2\pi} + \frac{1}{\pi}\sum_{j=1}^{\infty}\exp\{-\tau^\alpha j^\alpha\}\cos(j[\theta - \mu])\Big\}d\theta.$$

The change of variables from Cartesian to polar coordinates $\lambda_1 = \rho\cos\psi$, $\lambda_2 = \rho\sin\psi$, allows to re-express the jth integrand in the sum above as

$$\frac{1}{\pi}\exp\{-\tau^\alpha j^\alpha\}\int_0^{2\pi}\exp\{\rho(\cos\psi\cos\theta + \sin\psi\sin\theta)\}\cos(j[\theta - \mu])d\theta.$$

By re-expressing the exponent inside the integral as $\rho\cos(\theta - \psi)$, the change of variable of integration $\beta = \theta - \psi$ allows us to identify directly the terms

$$I_j(\rho) = \pi^{-1}\int_0^{\pi}\exp\{\rho\cos\theta\}\cos(j\theta)d\theta \text{ and } \int_0^{\pi}\exp\{\rho\sin\theta\}\cos(j\theta)d\theta = 0,$$

so that we can eventually find

$$M(\lambda) = I_0(||\lambda||) + 2\sum_{j=1}^{\infty}\exp\{-\tau^\alpha j^\alpha\}\cos\{j[\mu - \arg\{\lambda_1 + \iota\lambda_2\}]\}I_j(||\lambda||)$$

and (7) follows. □

PROOF OF PROPOSITION 3.1. The standard multivariate Central Limit Theorem applied to the i.i.d. unit vectors $(\cos\theta_i, \sin\theta_i)^{\mathrm{T}}$, $i = 1, \ldots, n$, tells us that

$$\frac{1}{\sqrt{n}}\begin{pmatrix} \sum_{i=1}^n(\cos\theta_i - \mathrm{E}[\cos\theta_1]) \\ \sum_{i=1}^n(\sin\theta_i - \mathrm{E}[\sin\theta_1]) \end{pmatrix} \xrightarrow{\mathcal{D}} N(0,\Sigma),$$

where

$$\Sigma = \begin{pmatrix} \sigma_{CC} & \sigma_{CS} \\ \sigma_{SC} & \sigma_{SS} \end{pmatrix}$$

is the covariance matrix. The elements of Σ can be identified via the two first trigonometric moments of a WSαS distribution, given by (11) with $j = 1$ and $j = 2$. If we define $\xi = (\xi_C, \xi_S)^{\mathrm{T}} = (\mathrm{E}[\cos\theta_1], \mathrm{E}[\sin\theta_1])^{\mathrm{T}}$, we can identify: $\xi_C = \rho\cos\mu$, $\xi_S = \rho\sin\mu$, $\sigma_{CC} = 1/2 + \rho^{2^\alpha}\cos(2\mu)/2 - \rho^2\cos^2\mu$, $\sigma_{CS} = \sigma_{SC} = \rho^{2^\alpha}\sin(2\mu)/2 - \rho^2\cos\mu\sin\mu$, and $\omega_{SS} = 1/2 - \rho^{2^\alpha}\cos(2\mu)/2 - \rho^2\sin^2\mu$. The delta-method (see e.g. C. R. Rao, 1972, p. 388), as applied to the ratio $\sum_{i=1}^n \sin\theta_i / \sum_{i=1}^n \cos\theta_i$, leads to the asymptotic variance

$$\zeta^2 = \frac{1}{\xi_C^2}\Big\{\sigma_{CC}\Big(\frac{\xi_S}{\xi_C}\Big)^2 + \sigma_{SS} - 2\sigma_{CS}\frac{\xi_S}{\xi_C}\Big\}. \tag{17}$$

The final expression for ζ^2 is obtained after inserting the values of ξ and Σ into (17), with some additional simplifications. □

PROOF OF PROPOSITION 3.2. Let us denote by $f_{\hat{\mu},R}$ the joint density of $(\hat{\mu}, R_n)$. The conditional density of $\hat{\mu}$ given $R_n = r$, at point $\omega \in [0, 2\pi)$, is given by

$$\frac{f_{\hat{\mu},R}(\omega, r)}{f_R(r)} = \frac{f_{C,S}(r\cos\omega, r\sin\omega)r}{\int_0^{2\pi} f_{\hat{\mu},R}(\nu, r)d\nu} = \frac{f_{C,S}(r\cos\omega, r\sin\omega)}{\int_0^{2\pi} f_{C,S}(r\cos\nu, r\sin\nu)d\nu}. \tag{18}$$

Integrating this last expression from 0 to $\theta \in [0, 2\pi)$, leads to the probability $\mathrm{P}[0 \le \hat{\mu} < \theta \mid R_n = r]$. Replacing $f_{C,S}$ by its joint saddlepoint approximation $g_{C,S}$ in both the numerator and the denominator of (18), and making the change of variable of integration in order to integrate in the cumulant generating function domain, leads to the final result. □

References

BARNDORFF-NIELSEN, O.E. and COX, D.R. (1989). *Asymptotic Techniques for Use in Statistics*. Chapman and Hall, London.

DANIELS, H.E. (1954). Saddlepoint approximations in statistics. *Ann. Math. Statist.*, **25**, 631-650.

EFRON, B. (1982). Better bootstrap confidence intervals. *J. Amer. Statist. Assoc.*, **82**, 171-185.

FAMA, E. and ROLL, R. (1968). Some properties of symmetric stable distributions. *J. Amer. Statist. Assoc.*, **63**, 817-836.

FAMA, E. and ROLL, R. (1971). Parameter estimates for symmetric stable distributions. *J. Amer. Statist. Assoc.*, **66**, 331-338.

FELLER, W. (1971). *An Introduction to Probability Theory and its Applications, Vol. 2.* Wiley, New York.

FIELD, C. (1982). Small sample asymptotic expansions for multivariate M-estimates. *Ann. Statist.*, **10**, 672-689.

FIELD, C. A. and RONCHETTI, E. (1990). *Small Sample Asymptotics*, IMS Lecture Notes-Monograph Series, Vol. 13. Institute of Mathematical Statistics, Hayward, CA.

FIELD, C.A. and TINGLEY, M.A. (1997). Small sample asymptotics: applications in robustness. In *Handbook of Statistics, Vol. 15*, Maddala and C.R. Rao, eds., North Holland, Amsterdam, 513-536.

FISHER, N.I. (1993). *Statistical Analysis of Circular Data*, Cambridge University Press, Cambridge.

GATTO, R. (2000). Multivariate saddlepoint test for the wrapped normal model. *J. Statist. Comput. Simulation*, **65**, 271-285.

GATTO, R. and RONCHETTI, E. (1996). General saddlepoint approximations of marginal densities and tail probabilities. *J. Amer. Statist. Assoc.*, **91**, 666-673.

GREENWOOD, J.A. and DURAND, D. (1955). The distribution of length and components of the sum of n random unit vectors. *Ann. Math. Statist.*, **26**, 233-246.

HALL, P. (1992). *The Bootstrap and Edgeworth Expansion*, Springer-Verlag, New York.

HAMPEL, F.R., RONCHETTI, E.M., ROUSSEEUW, P.J. and STAHEL, W.A. (1986). *Robust Statistics: The Approach Based on Influence Functions*, Wiley, New York.

JAMMALAMADAKA, S.R. and SENGUPTA, A. (2001). *Topics in Circular Statistics.* World Scientific, Singapore.

LÉVY, P. (1939). L'addition des variables aléatoires définies sur une circonférence. *Bull. Soc. Math. France*, **67**, 1-41.

LUGANNANI, R. and RICE, S. (1980). Saddle point approximation for the distribution of the sum of independent random variables. *Adv. Appl. Probab.*, **12**, 475-490.

MARDIA, K.V. (1972), *Statistics of Directional Data.* Academic Press, New York.

MARDIA, K.V. and JUPP, P.E. (2000). *Directional Statistics.* Wiley, New York.

RAO, C.R. (1972). *Linear Statistical Inference.* Wiley, New York.

SAMORODNITSKY, G. and TAQQU, M.S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, London.

SENGUPTA, S. and RAO, J.S. (1966). Statistical analysis of cross-bedding azimuths from the Kamthi formation around Bheemaram, Pranhita-Godavari valley. *Sankhyā Ser. B*, **28**, 165-174.

SKOVGAARD, I.M. (1987). Saddlepoint expansions for conditional distributions. *J. Appl. Probab.*, **24**, 875-887.

WANG, S. (1990). Saddlepoint approximations in resampling analysis. *Ann. Inst. Statist. Math.*, **42**, 115-131.

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